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Duality In Matrix Theory And Three Dimensional Mirror Symmetry

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Certain limits of the duality between M-theory on T^5/Z_2 and IIB on K3 are analyzed in Matrix theory. The correspondence between M-theory five-branes and ALE backgrounds is realized as three dimensional mirror symmetry. Non-critical strings dual to open membranes are explicitly described as gauge theory excitations. We also comment on Type IIA on K3 and the appearance of gauge symmetry enhancement at special points in the moduli space.

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1. Introduction

Recently, it has become clear that the non-perturbative formulation of M-theory proposed by Banks, Fischler, Shenker and Susskind [1] captures many essential aspects of conventional string physics. Various tests of this theory include the derivation of elementary strings and their interactions [2,3,4], construction and scattering of solitonic states [1,5,6,7], compactifications on tori and K3 surfaces [8,9,10,11,12,13,14,15,16,17], and finally realizations of standard dualities [8,16,17].

The purpose of the present paper is to study the Matrix theory description of the M-theory on $T^5/Z_2 \leftrightarrow$ IIB on $K3$ duality [18]. Since compactification of Matrix theory on high dimensional tori involves certain subtleties [14,15] we restrict to special limits in the moduli space where both models can be realized as three dimensional gauge theories. In M-theory this corresponds to a degeneration of the five-torus of the form $T^5 \simeq T^2 \times R^3$ while in IIB theory we consider non-compact K3 surfaces described by ALE gravitational instantons. It turns out that this framework allows a realization of the conventional ALE $\leftrightarrow$ five-brane duality [19] as three dimensional mirror symmetry [20,21,22,23] as summarized below.

- A configuration of n five-branes away from the fixed point in $(T^2 \times R^3)/Z_2$ is described by an $\mathcal{N}=4$ $U(k)$ gauge theory with n fundamental hypermultiplets and one adjoint. The mirror pair is defined by an A_{n-1} quiver and describes IIB theory on an A_{n-1} ALE background. Non-critical Type IIB strings filling out an $SO(n+1,2,Z)$ multiplet are identified as excitations in the gauge theory. By mirror symmetry we obtain a gauge theory description of open membranes stretched between five-branes.

- A configuration of n five-branes near the fixed point is described by an $\mathcal{N}=4$ $Sp(k)$ gauge theory with n fundamental hypermultiplets and one anti-symmetric tensor. The mirror pair is defined by an D_n quiver and describes IIB theory on a D_n ALE background. Again, we identify the $SO(n+2,2,Z)$ multiplet of non-critical strings providing a mirror description of dual open membranes.

We also consider the Matrix realization of IIA theory on an ALE space emphasizing the phenomenon of gauge symmetry enhancement in the orbifold limit. The description of the relevant wrapped D2-brane states parallels that of Type IIB non-critical strings.

Section 2 is a brief review of certain aspects of the conventional theories introduced above. In section 3 we describe five-brane backgrounds in Matrix theory compactified on two tori. The realization of IIA and IIB models is developed in section 4. Section 5 consists of a detailed presentation of the mirror map between non-critical strings and open membranes.

2. Type IIA and Type IIB on K3 and M-theory on T^5/Z_2

The moduli space of Type IIB theory on K3 is [24,25,26,28] the Grassmanian of space-like five-planes in $R^{21,5}$ modulo discrete identifications

$$\mathcal{M} = SO(21,5,Z)\backslash SO(21,5)/(SO(21)\times SO(5)) \tag{2.1}$$

This can be equivalently regarded as the moduli space of even self-dual lattices $\Gamma^{21,5}$ representing the K3 cohomology lattice $\Gamma^{19,3}$ supplemented by two copies of the hyperbolic plane $\Gamma^{1,1}\oplus\Gamma^{1,1}$ corresponding to B and C moduli. The six-dimensional dynamics is characterized [27,28] by the occurrence of an $SO(21,5,Z)$ multiplet of non-critical BPS strings whose charges are vectors in $\Gamma^{21,5}$. The states represented by charge vectors in $\Gamma^{19,3}\subset\Gamma^{21,5}$ can be identified as wrapped D3-brane states with a BPS tension formula

$$T = \frac{1}{g_s}\sqrt{B^2+C^2+|\Omega|^2+J^2} \tag{2.2}$$

where Ω, J are the values of the holomorphic two-form and Kähler class on the cycle.

We will consider the theory locally near a blown-up ADE singularity embedded in a K3 surface of very large radius. The cohomology lattice spanned by the exceptional cycles is $\Gamma^{n-1,0}$ for A_{n-1} ALE and $\Gamma^{n,0}$ for D_n ALE. Accordingly, the non-critical strings will fill out $SO(n+1,2,Z)$ and $SO(n+2,2,Z)$ multiplets respectively.

Type IIA theory has a similar moduli space [24,26,29,30] parameterized by the quotient

$$\mathcal{M} = SO(20,4,Z)\backslash SO(20,4)/(SO(20)\times SO(4)) \tag{2.3}$$

The six-dimensional theory contains in this case BPS particles rather than strings, a subset of which can be identified with wrapped D2-brane states. The BPS mass formula is analogous

$$m = \frac{1}{g_s}\sqrt{B^2+|\Omega|^2+J^2} \tag{2.4}$$

According to [18] Type IIB theory on K3 is dual to M-theory on T^5/Z_2. Anomaly cancelation and magnetic charge conservation requires 16 M-theory five-branes in the background. The non-critical strings are dual [18,28] to open membranes stretching between five-branes [31,32]. In particular configurations with multiple five-branes are dual to ADE singularities in the K3 surface. As stated in the introduction we consider a particular limit of the moduli space in which the five-torus degenerates as $T^5\simeq T^2\times R^3$. This creates two distinct situations depending whether the five-branes coalesce at the fixed point or at a distant point in R^3. The relevant singularities are respectively A_{n-1} and D_n, where n is the number of branes.

3. Matrix on Tori and Five-Branes

It has been shown in [15] that compactification of Matrix theory on T^5/Z_2 can be described in terms of a new six-dimensional theory with eight supercharges modeled by heterotic five-branes at zero coupling. The theories studied there are related to $Spin(32)$ instantons of charge k but we will need a slight modification corresponding to $Spin(n)$ instantons. The low energy limit is a six-dimensional $\mathcal{N}=(1,0)$ $Sp(k)$ gauge theory with n fundamental hypermultiplets and one antisymmetric tensor. As emphasized in [14,15], compactification of the full theory on a generic five-torus does not have a well defined moduli space. However, in the case of interest here the space-time torus is $T^5 \simeq T^2 \times R^3$ so the dual torus degenerates to $\tilde{T}^2 \times T^3$ with T^3 shrinking to zero size. The low energy limit reduces to an $\mathcal{N}=4$ three-dimensional gauge theory on $R \times \tilde{T}^2$ with the same field content describing a configuration of n five-branes coming together at the fixed point in $T^2 \times R^3$. Note that the same effective description has been realized earlier in [12,13] starting from the original definition of Matrix theory as quantum mechanics of D0-branes. Also the construction outlined above incorporates automatically the fundamental hypermultiplets which describe longitudinal five-branes [5].

Moving away from the fixed point, the geometry is locally $T^2 \times R^3$ since the Z_2 projection relates tori at different points in R^3. A configuration of n coinciding five-branes will be described by a three-dimensional gauge theory with $U(k)$ gauge group, n hypermultiplets in the fundamental and one in the adjoint. In the picture of [12,13], this can be understood as a splitting of the original configuration of $2k$ D0-branes in two equal groups far apart from each other and decoupling of heavy open strings that stretch from one group to the other. The Z_2 projection simply relates the two separated groups so the gauge group is actually $U(k)$.

Further evidence supporting this picture comes from identifying the Coulomb branches of the gauge theories with five-brane backgrounds. We will outline the details only for $U(k)$ theories, the $Sp(k)$ being analogous. The parameters of the gauge theory are [10]

$$\frac{1}{g^2} = \frac{L_1 L_2}{(2\pi)^2 R}, \qquad \Sigma_m = \frac{(2\pi)^2 l_{11}^3}{R L_m}, \qquad m = 1,\, 2 \tag{3.1}$$

where L_m are the circumferences of the space-time torus. As argued in [10], Type IIB is obtained in the limit $L_m \to 0$ with $L_2/L_1 << 1$ fixed. In this regime the dual torus approaches the decompactification limit at fixed $\Sigma_1/\Sigma_2 << 1$.

The result can be viewed as either IIB theory on a circle of circumference

$$L_Y = (2\pi)^3 \frac{l_{11}^3}{L_1 L_2} \tag{3.2}$$

or, by T-duality, IIA theory on a circle of circumference L_2 [10]. The string scale and the Type IIB coupling constant are given by

$$\alpha' = \frac{2\pi l_{11}^3}{L_2}, \qquad g_s = \frac{L_2}{L_1}. \tag{3.3}$$

After dualizing the low energy photons, the $U(k)$ reduces essentially to k identical copies of a $U(1)$ gauge theory with the same field content. Thus it suffices to study the Coulomb branch of the latter. In vector multiplet variables $(\vec{\phi}, A_\mu)$ the one-loop corrections are exact [33,34] and consist of the metric

$$ds^2 = \left(\frac{1}{g^2} + \sum_{i=0}^{n-1} \frac{1}{|\vec{\phi} - \vec{m}_i|} \right) d\vec{\phi} \cdot d\vec{\phi}. \tag{3.4}$$

and a Chern-Simons term [1]

$$\sum_{i=0}^{n-1} \epsilon^{\lambda\mu\nu} \vec{\omega}_i \cdot \partial_\lambda \vec{\phi} F_{\mu\nu} \tag{3.5}$$

where ω_i are standard Dirac monopole vector potentials and $\vec{m}_i$ are hypermultiplet bare masses. Since the dual torus $\tilde{T}^2$ grows as $R \times S^1$ the theory is effectively compactified on a circle of very large radius. Dimensional reduction of the three dimensional Coulomb branch [2] yields an $\mathcal{N} = (4,4)$ two dimensional sigma-model with torsion

$$\frac{1}{2\pi\alpha'} \int d\sigma_2 d\tau \left\{ \left(1 + \sum_{i=0}^{n-1} \frac{2\pi\alpha'}{R_1} \frac{1}{|\vec{r} - \vec{r}_i|} \right) \left(d\vec{r}^2 + R_1^2 d\theta_1^2 \right) + 2\pi\alpha' \sum_{i=0}^{n-1} \epsilon^{\mu\nu} \partial_\mu \vec{\omega}_i \cdot \partial_\nu \vec{r} \right\}. \tag{3.6}$$

where the conversion between space-time and gauge theory parameters is given by

$$\vec{r} = \frac{2\pi l_{11}^3}{R_2} \vec{\phi}, \qquad \vec{r}_i = \frac{2\pi l_{11}^3}{R_2} \vec{m}_i. \tag{3.7}$$

It is easy to check that this is exactly the symmetric monopole solution of [35], thus the gauge theory indeed describes five-brane backgrounds as claimed before. A similar check has been performed in two dimensional context in [36].

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[1] Actually this term is obtained from that of [33,34] by an integration by parts.

[2] Strictly speaking, this procedure is valid only far out on flat directions where the radius of the circle is much larger than the length scale $\frac{1}{|\vec{\phi}|}$ and it should break down in a central region of the moduli space. However, this region can be made arbitrarily small by making the radius of the circle arbitrarily big. This is the regime in which the three dimensional description is valid.

4. Matrix description of string theory on ALE spaces

The standard approach to the IMF definition of M-theory on non-compact ALE backgrounds [9,11] is the quantum mechanics [3] of D0-branes on a C^2/Γ orbifold [38]. However it has been argued recently, [14,15,17] that a complete treatment of compact K3 surfaces involves new six-dimensional theories compactified on the dual K3 times a circle. In particular certain degrees of freedom related to the exceptional cycles of a blown-up singularity are absent from the usual construction. Since at the present stage it is not clear how to make the latter description explicit, we will consider the first approach. Even if this should be regarded as an approximate description of the theory, it will turn out that it captures the description of wrapped D-brane states.

4.1. IIA on ALE

As argued in [9,11] this theory can be realized in terms of a two dimensional $\mathcal{N} = (4,4)$ gauge theory on $S^1 \times R$ with field content specified by a quiver diagram. The parameters of the gauge theory have been determined in [10]

$$\frac{1}{g^2} = \frac{\Sigma L^2}{(2\pi)^2 R}, \qquad \Sigma L = \frac{(2\pi)^2 l_{11}^3}{R}. \tag{4.1}$$

In addition we introduce Fayet-Iliopoulos parameters which can be related to the blow-up modes present in the original quantum mechanics by

$$\vec{\zeta}_{gauge} = \frac{R^2}{(2\pi)^2 l_{11}^6} \vec{\zeta}_{string}. \tag{4.2}$$

This is just the usual conversion between length in space-time and mass in gauge theory. From now on the gauge theory parameters will be denoted simply by $\vec{\zeta}$.

• A_{n-1} *ALE*. The relevant gauge dynamics is related to the Higgs branch described by a hyper-Kähler quotient construction [39,40]

$$\begin{aligned}
\mu_0^R &\equiv X_{01}X_{01}^\dagger - X_{10}^\dagger X_{10} + X_{0,n-1}X_{0,n-1}^\dagger - X_{n-1,0}^\dagger X_{n-1,0} = \zeta_0^R\, I_{k\times k} \\
\mu_0^C &\equiv X_{01}X_{10} - X_{0,n-1}X_{n-1,0} = \zeta_0^C\, I_{k\times k} \\
&\;\;\vdots \\
\mu_{n-1}^R &\equiv X_{n-1,0}X_{n-1,0}^\dagger - X_{0,n-1}^\dagger X_{0,n-1} + X_{n-1,n-2}X_{n-1,n-2}^\dagger \\
&\quad - X_{n-2,n-1}^\dagger X_{n-2,n-1} = \zeta_{n-1}^R\, I_{k\times k} \\
\mu_{n-1}^C &\equiv X_{n-1,0}X_{0,n-1} - X_{n-1,n-2}X_{n-2,n-1} = \zeta_{n-1}^C\, I_{k\times k}
\end{aligned} \tag{4.3}$$

where the X's are in one-to-one correspondence with the links of the following quiver diagram

[3] This quantum mechanics was first considered in [37].

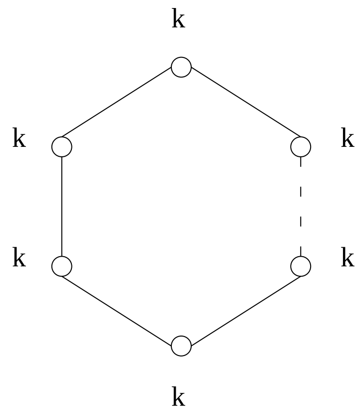

Fig. 1: A_{n-1} quiver diagram.

and the Fayet-Iliopoulos parameters are constrained by

$$\sum_{i=0}^{n-1} \vec{\zeta}_i = 0. \tag{4.4}$$

The moduli space of solutions modulo gauge transformations is isomorphic to the symmetric product $Sym^k(\widetilde{X}_{A_{n-1}})$ of resolved ALE varieties of A_{n-1} type. There is no Coulomb branch and the gauge group is generically broken to the diagonal torus $U(1)^k_D \subset U(k)^n$ along the hypermultiplet flat directions. Note that there is a one to one correspondence between vertices of the extended Dynkin diagram and homology two cycles γ_i of the blown-up ALE satisfying the linear constraint

$$\sum_{i=0}^{n-1} \gamma_i = 0. \tag{4.5}$$

The schematic form of the action obtained by reduction from six dimensions reads

$$\begin{aligned} \frac{1}{g^2} \int dt d\sigma \Big\{ & \frac{1}{2n} Tr\Big(F_0^2 + F_1^2 + \ldots + F_{n-1}^2 + \vec{D}_0^2 + \vec{D}_1^2 + \ldots + \vec{D}_{n-1}^2 \Big) + \\ & Tr\Big(\vec{D}_0 \cdot (\vec{\mu}_0 - \vec{\zeta}_0\, I_{k\times k}) + \vec{D}_1 \cdot (\vec{\mu}_1 - \vec{\zeta}_1\, I_{k\times k}) + \ldots \\ & \vec{D}_{n-1} \cdot (\vec{\mu}_{n-1} - \vec{\zeta}_{n-1}\, I_{k\times k}) \Big) \Big\} \end{aligned} \tag{4.6}$$

where $\vec{D}$ are triplets of auxiliary fields in the six-dimensional vector multiplet and $\vec{\mu}$ are the moment maps introduced in (4.3). The factor $1/n$ multiplying the gauge kinetic terms is inherited from the original quantum mechanics where a single D0-brane moving on the ALE is described by n identical images under the orbifold group Z_n. Thus the normalization is fixed such that summing over all images yields the correct kinetic term.

Different states in the resulting Type IIA theory can be identified with gauge theory excitations as follows. D0-branes propagating freely on the ALE background can be described by turning on t'Hooft fluxes $\frac{1}{kn} I_{k\times k}$ in each of the n factors of the overall diagonal $U(1) \subset U(k)^n$. The energy of a single quantum of flux is computed by summing over all n fluxes and the correct scaling behavior is ensured by the factor $1/n$ mentioned above [4]

$$E_E = \frac{R}{2k}\left(\frac{2\pi}{L}\right)^2 = \frac{1}{2p_{11}}\left(\frac{1}{g_s\sqrt{\alpha'}}\right)^2. \tag{4.7}$$

It matches the light cone energy of a single D0-brane carrying k units of longitudinal momentum. The string theory parameters present in this formula characterize the newly defined theory and are given by

$$\alpha' = \frac{2\pi l_{11}^3}{L}, \qquad g_s^2 = \frac{1}{(2\pi)^3}\frac{L^3}{l_{11}^3}. \tag{4.8}$$

As stated in section one the theory is expected to contain more general BPS states with charge vectors in the cohomology lattice $\Gamma^{n-1,0}$. In conventional string theory these have been identified as fractional D0-branes described by a single image of the orbifold gauge group [9,41,42]. The hypermultiplet degrees of freedom corresponding to motion on the ALE are projected out. Therefore the object is stuck at the fixed point but free to move in transverse directions.

In Matrix gauge theory these states can be described as positive energy excitations of the hypermultiplets away from the Higgs branch [9]. In order to identify wrapped states on the ith cycle γ_i we propose to set the expectation values of the hypermultiplets charged under $U(k)_i$ to zero [5]. We will refer to this mechanism as amputation of the quiver diagram (see fig. 2). This results in a positive energy field configuration sitting at a stationary point of the potential. The scalars in the $U(k)_i$ vector multiplet will parameterize a new "Coulomb" branch of *meta-stable flat directions*.

To prove that the energy of these excitations has the correct scaling behavior, note that the gauge group is $U(k) = (SU(k)\times U(1))/Z_k$ and that we have matter in fundamental representation which is acted faithfully by the center Z_k. Therefore by normalizing the $SU(k)$ charges to be integers, the $U(1)$ charges have to be quantized in $1/k$ [6]. This factor

[4] We thank T. Banks for clarifying explanations on this point.

[5] We thank M. Douglas for suggesting this.

[6] We thank N. Seiberg and S. Shenker for explaining this to us.

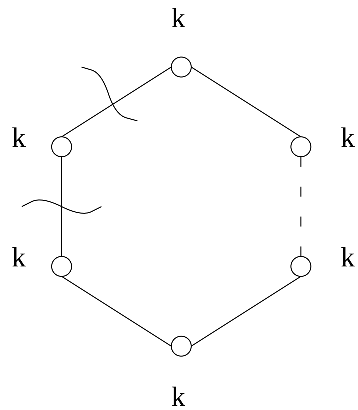

Fig. 2: Amputation of adjacent legs in an A_{n-1} quiver diagram. This describes a wrapped state on the isolated cycle.

will affect all couplings of the diagonal $U(1) \subset U(k)$ with fundamental hypermultiplets. Integrating out the diagonal $\vec{D}$ terms in (4.6) results in an energy

$$\Delta E = \frac{1}{2}\frac{nR}{k}\frac{1}{(2\pi)^2 l_{11}^6}\{|\vec{\zeta}_i|^2 + \frac{1}{n-1}|\sum_{j\neq i}\vec{\zeta}_j|^2\} \tag{4.9}$$

In the IMF this corresponds to an overlap of two states with mass and longitudinal momentum given by

$$\begin{aligned} M^2 &= \frac{|\vec{\zeta}_i|^2}{(2\pi)^2 g_s^2 \alpha'^3} \qquad & p_{11} &= \frac{1}{n}\frac{k}{R} \\ M'^2 &= \frac{|\sum_{j\neq i}\vec{\zeta}_j|^2}{(2\pi)^2 g_s^2 \alpha'^3} \qquad & p'_{11} &= \frac{n-1}{n}\frac{k}{R} \end{aligned} . \tag{4.10}$$

Note that (4.4) implies that $M = M'$. We interpret this as a configuration of two D2-branes with opposite charges wrapped on the cycle γ_i and the complementary cycle $\sum_{j\neq i}\gamma_j = -\gamma_i$. Therefore these states can be viewed as individual stable particles only if they are separated far apart in transverse directions. In principle they can come together and annihilate into uncharged states [7]. A detailed study of these issues will appear elsewhere [43]. The B-field on cycles is zero, thus these states become massless in the orbifold limit. This should be contrasted with the situation in the original IIA theory [9,41].

This procedure can be generalized to obtain bound states of D2-branes wrapped on linear combination of cycles $\sum_{i=0}^{p-1}\gamma_i$. The links of the quiver diagram are cut off such that one is left with two separate groups of p and respectively $n-p$ interconnected vertices.

[7] We thank M. Douglas for ample explanations on these issues.

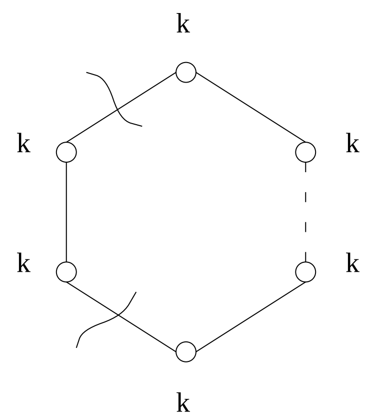

Fig. 3: Amputation of non-adjacent legs in an A_{n-1} quiver diagram. This describes a bound state of wrapped states on the two isolated cycles.

The case $p = 2$ is represented in fig. 3. In this case the energy is

$$\Delta E = \frac{1}{2}\frac{nR}{k}\frac{1}{(2\pi)^2 l_{11}^6}\left\{\frac{1}{p}|\sum_{i=0}^{p-1}\vec{\zeta}_i|^2 + \frac{1}{n-p}|\sum_{i=p}^{n}\vec{\zeta}_i|^2\right\}. \tag{4.11}$$

As before, we interpret this configuration as two wrapped states with opposite charges. The masses and longitudinal momenta read

$$\begin{aligned} M^2 &= \frac{|\sum_{i=0}^{p-1}\vec{\zeta}_i|^2}{(2\pi)^2 g_s^2 \alpha'^3} \qquad & p_{11} &= \frac{p}{n}\frac{k}{R} \\ M'^2 &= \frac{|\sum_{i=p}^{n}\vec{\zeta}_i|^2}{(2\pi)^2 g_s^2 \alpha'^3} \qquad & p'_{11} &= \frac{(n-p)}{n}\frac{k}{R} \end{aligned}. \tag{4.12}$$

Therefore we obtain bound states in $1-1$ correspondence with the roots of the Dynkin diagram filling out the expected non-Abelian vector multiplet in the orbifold limit.

- D_n *ALE*. These models can be described similarly to the A_{n-1} case with minor differences. The gauge theory is defined by the regular D_n quiver diagram. The hyper-Kähler moment maps defining the Higgs branch $Sym^k(\tilde{X}_{D_n})$ are similar to (4.3). The action is essentially obtained from (4.6) by replacing the factor n with the dual Coxeter number $c_2 = 2(n-1)$.

String theory states can be described similarly to the A_{n-1} case. D0-branes propagating freely on the ALE are identified with t'Hooft fluxes $\frac{1}{c_2 k}$ in the overall diagonal $U(1) \subset U(k)^4 \times U(2k)^{n-3}$.

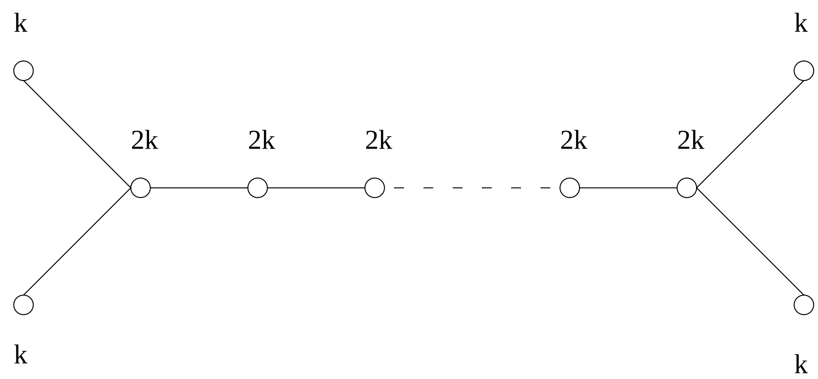

Fig. 4: D_n quiver diagram.

Excitations corresponding to wrapped D2-branes can be obtained by exciting the hypermultiplets as before. In this case we must distinguish between cycles represented by outer and respectively inner vertices of the extended Dynkin diagram. Amputation of the D_n quiver will generally result in configurations of several wrapped states whose charges sum to zero. The IMF mass and longitudinal momenta can be computed similarly to (4.9).

4.2. IIB on ALE

Type IIB on an $S^1 \times$ ALE background can be constructed as the three dimensional counter-part of the quiver gauge theories studied in the previous section. More precisely compactification of the orbifold quantum mechanics on a two torus yields the three dimensional $\mathcal{N} = 4$ quiver gauge theories studied extensively in [20,21,22,23]. The parameters of the gauge theory as well as those of the derived string theory have been specified in (3.1)-(3.3) and (4.2). We consider again two cases:

- A_{n-1} *ALE*. The model is defined by the quiver in fig. 1. The gauge dynamics on the Higgs branch is identical to the two dimensional theory but the string interpretation of the excitations is different. Electric fluxes in the overall diagonal $U(1) \subset U(k)^n$ can be identified with IIB fundamental or D-strings propagating on the ALE background. The energy of a single quanta of flux in direction $m = 1, 2$ is:

$$E_E = \frac{R}{2k} \left(\frac{L_Y L_m}{(2\pi)^2 g_s \alpha'^{3/2}} \right)^2 . \tag{4.13}$$

For $m = 1$ this matches the IMF energy of a D-string wrapped on S^1_Y while for $m = 2$ this matches the IMF energy of a wrapped fundamental string as in [10]. Again the fact that these strings are free to propagate on the ALE is consistent with the diagonal $U(1)$ being generically unbroken on the Higgs branch.

As explained in section one IIB theory on an ALE space has in addition an $SO(2, n+1, Z)$ multiplet of non-critical BPS strings whose charge vectors live in the lattice $\Gamma^{0,n-1} \oplus \Gamma^{2,2}$. The first factor is the homology lattice of the resolved ALE space and the corresponding non-critical strings are actually D3-branes wrapped on the homology two cycles. These are T-dual to wrapped D2-branes in Type IIA theory, thus we know how to identify them as meta-stable positive energy excitations in the gauge theory. The mechanism is the same and we only need to check that the $\vec{D}$ term energy has the correct scaling behavior. Amputation of the quiver diagram as in fig. 2 results in a configuration with energy

$$\Delta E = \frac{nR}{2k} \frac{1}{(2\pi)^2 l_{11}^6} \left\{ |\vec{\zeta}_i|^2 + \frac{1}{n-1} |\sum_{j \neq i} \vec{\zeta}_j|^2 \right\} \tag{4.14}$$

In the IMF this corresponds to an overlap of two states with masses and longitudinal momenta given by

$$\begin{aligned} M^2 &= \frac{|\vec{\zeta}_i|^2 L_Y^2}{(2\pi)^4 g_s^2 \alpha'^4} & p_{11} &= \frac{1}{n}\frac{k}{R} \\ M'^2 &= \frac{|\sum_{j\neq i} \vec{\zeta}_j|^2 L_Y^2}{(2\pi)^4 g_s^2 \alpha'^4} & p'_{11} &= \frac{n-1}{n}\frac{k}{R} \end{aligned} . \tag{4.15}$$

As before, we interpret this as a metastable configuration of D3-branes with opposite charges wrapped on the cycle γ_i and the extra circle S_Y^1. In order to identify individual stable states they must be given a large separation in the transverse directions.

We can also identify non-critical strings corresponding to non-simple roots in the homology lattice of the ALE as bound states of D3-branes wrapped on linear combinations of cycles. The correct scaling behavior can be shown in a calculation identical to (4.11). However, there are more general non-critical strings whose charge vectors have non-zero components in the second factor $\Gamma^{2,2}$ that carry charge (p, q) under the bulk two form potentials B, C. In conventional string theory these may be identified as non-marginal bound states of wrapped D3-branes and (p, q) strings whose mass is given by:

$$M^2 = T_3^2 |\vec{\zeta}_i|^2 L_Y^2 + T_{(p,q)}^2 L_Y^2. \tag{4.16}$$

Realization of these states in Matrix picture is non-trivial and involves crucially the new branch of meta-stable flat directions associated to the amputated quiver diagram in fig. 2. Since the expectation values of the hypermultiplets charged under $U(k)_i$ have been set to zero, the full $U(k)_i$ is un-higgsed at the origin. The meta-stable flat directions are parameterized by scalars in the vector multiplet thus there will be generically k low energy mass-less photons $U(1)^k$. We stress that these are not the same as the generic mass-less photons on the Higgs branch.

Turning on a t'Hooft flux $\frac{1}{k} I_{k\times k}$ in the diagonal $U(1) \subset U(k)_i$ along Σ_m increases the energy of an individual wrapped state on the cycle γ_i to

$$\Delta E = \frac{nR}{2k} \frac{|\vec{\zeta}_i|^2 L_Y^2}{(2\pi)^8 g_s^2 \alpha'^4} + \frac{nR}{2k} \left(\frac{L_Y L_m}{(2\pi)^2 g_s \alpha'^{3/2}} \right)^2 \tag{4.17}$$

which results in an IMF mass

$$M^2 = \frac{|\vec{\zeta}_i|^2 L_Y^2}{(2\pi)^8 g_s^2 \alpha'^4} + \left(\frac{L_Y L_m}{(2\pi)^2 g_s \alpha^{3/2}} \right)^2 . \tag{4.18}$$

Taking into account (4.13) and (4.14), it is clear that this result is in agreement with the one expected from string theory. It is interesting to trace the origin of the factor of n in the energy of the flux quanta. This follows from the fact that in the original quantum mechanics the gauge kinetic term for a single image is $\frac{1}{2ng_s} F^2$ as opposed to the gauge kinetic term for the diagonal $U(1)$. We conclude that all non-critical strings in the $SO(2, n+1, Z)$ multiplet can be described in Matrix theory.

- D_n *ALE*. The analysis is very similar to the A_{n-1} case and it will not be repeated here. As in the previous section we must differentiate between outer and inner cycles. The states wrapped on inner cycles carry $\frac{2k}{c_2}$ units of longitudinal momentum while those wrapped on outer cycles carry $\frac{k}{c_2}$ units.

5. Duality and Mirror Symmetry

The results obtained in the previous sections can be best summarized in the following diagrams

$$\begin{array}{ccc} \text{Type IIB on } A_{n-1} \text{ ALE} & \longleftrightarrow & \text{M-theory on } T^2 \times R^3 \text{ with } n \text{ five-branes} \\ \wr & & \wr \\ 3D\ A_{n-1} \text{ quiver gauge theory} & \longleftrightarrow & 3D\ U(k) \text{ gauge theory with } n \text{ hypermultiplets} \end{array}$$

$$\begin{array}{ccc} \text{Type IIB on } D_n \text{ ALE} & \longleftrightarrow & \text{M-theory on } (T^2 \times R^3)/Z_2 \text{ with } n \text{ five-branes} \\ \wr & & \wr \\ 3D\ D_n \text{ quiver gauge theory} & \longleftrightarrow & 3D\ Sp(k) \text{ gauge theory with } n \text{ hypermultiplets} \end{array}$$

The upper arrow denotes conventional duality, $\wr$ stands for Matrix theory realization while the lower arrow is at this stage the missing link in the chain. It turns out that the diagram is in fact closed by the *three dimensional mirror symmetry* discovered recently

in [20] and studied further in [21,22,23]. The pairs of gauge theories describing the above Matrix model backgrounds are related by electric-magnetic duality which maps in a precise manner the Higgs/Coulomb branch of the first theory to the Coulomb/Higgs branch of the second. The FI parameters of the quiver theories are mapped to bare hypermultiplet masses. Taking into account the geometric interpretation of the two sets of parameters we have derived a Matrix theory realization of the duality between coalescing five-branes and ADE singularities.

Moreover, in M-theory the non-critical strings are membranes stretching between five-branes which become tensionless as the five-branes approach each other. Therefore we have also obtained a Matrix-theory description of open membranes, although from a dual point of view. Pictorially, the correspondence can be represented as follows in the A_{n-1} case.

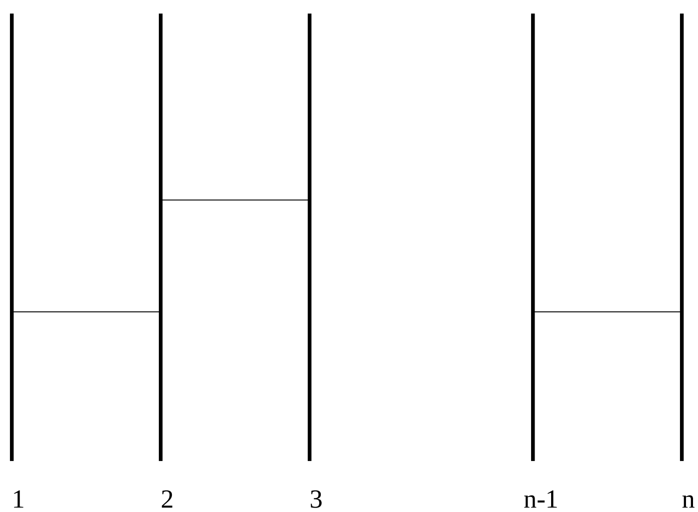

Fig. 5: The membrane configuration dual to an A_{n-1} quiver.

The simple roots of an A_{n-1} Dynkin diagram can be represented in terms of an orthonormal basis $\vec{e}_i$ in an n dimensional vector space as:

$$\vec{\alpha}_1 = \vec{e}_1 - \vec{e}_2, \ \ldots, \vec{\alpha}_{n-1} = \vec{e}_{n-1} - \vec{e}_n.$$

We can assign a five-brane to each vector such that a pair of consecutive five-branes determines a simple root. Open membranes stretching between five-branes are represented by horizontal line segments. The positions of the five-branes are related to the sizes of the exceptional cycles by the mirror map [20,21]:

$$\vec{m}_i = \sum_{l=0}^{i} \vec{\zeta}_l$$

such that the tensions of open membranes match the masses of wrapped D3-branes.

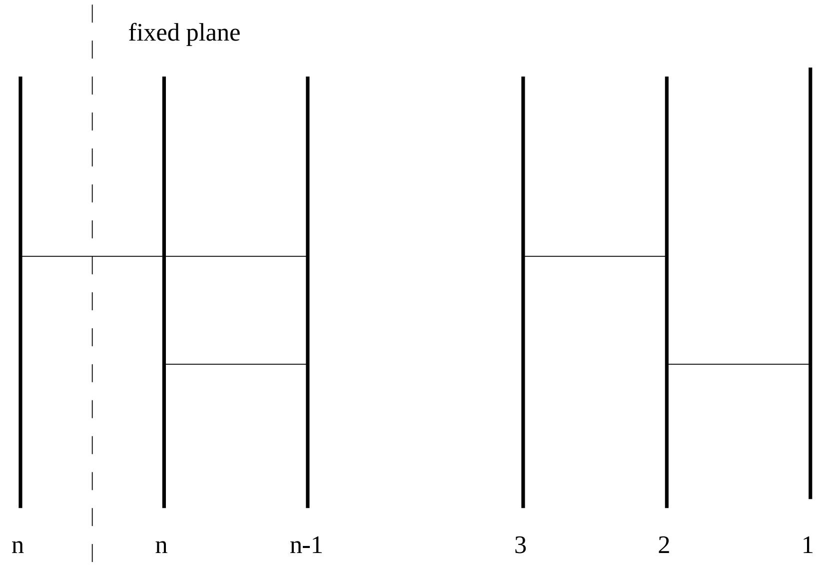

Fig. 6: The membrane configuration dual to a D_n quiver.

Similar considerations apply in the D_n case. The simple roots of a D_n Dynkin diagram can be represented in terms of the basis vectors as:

$$\vec{\beta}_1 = \vec{e}_1 - \vec{e}_2, \ \ldots, \vec{\beta}_{n-1} = \vec{e}_{n-1} - \vec{e}_n, \ \vec{\beta}_n = \vec{e}_{n-1} + \vec{e}_n.$$

Note that the nth root is the reflection of the $(n-1)$th in the hyperplane $\vec{e}_n^{\perp}$. When assigning as before a five-brane to each basis vector the last root will be determined by the $(n-1)$th five-brane and the Z_2 image of the nth one. The mirror map is in this case:

$$\vec{m}_n = \vec{\zeta}_n - \vec{\zeta}_{n-1},$$

$$\vec{m}_{n-1} = \vec{\zeta}_n + \vec{\zeta}_{n-1},$$

$$\vec{m}_{n-2} = 2\vec{\zeta}_{n-2} + \vec{\zeta}_n + \vec{\zeta}_{n-1},$$

$$\vdots$$

$$\vec{m}_1 = 2\vec{\zeta}_1 + 2\vec{\zeta}_2 + \ldots + 2\vec{\zeta}_{n-2} + \vec{\zeta}_n + \vec{\zeta}_{n-1}.$$

D3-branes wrapped on the ith cycle correspond to open membranes stretched between the $(i, i+1)$ five-branes except for $i = n$ when they correspond to membranes stretched between the $(n-1)$th five-brane and the Z_2 image of the nth five-brane.

6. Discussion

In this paper, further evidence for the validity of Matrix theory as the IMF description of eleven dimensional M-theory is presented. Certain local aspects of the conventional duality $\mathrm{M}/(T^5/Z_2) \sim \mathrm{IIB}/K3$ are realized in Matrix theory in terms of three dimensional mirror symmetry.

We also analyze ALE backgrounds in the Matrix theory definition of Type IIA and Type IIB string theory. The expected multiplets of wrapped states are identified as gauge theory excitations. The analysis reveals their bound state structure in precise agreement with conventional string theory. Moreover it is shown that mirror symmetry maps the description of non-critical IIB strings to open membranes stretching between M-theory five-branes.

The SYM description used extensively above is only valid in special regimes. A full analysis would require detailed information on the six dimensional theories introduced in [14,15]. In section 3 we argued that the gauge theories describing five-brane backgrounds can be regarded as low energy limits of the full six dimensional $(1,0)$ theory based on $Spin(32)$ instantons. It would be very interesting to repeat the analysis for quiver gauge theories and the $(2,0)$ six dimensional string theory on $K3 \times S^1$.

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References

[1] T. Banks, W. Fischler, S.H. Shenker, L.Susskind, "M-Theory As a Matrix Model: A Conjecture", Phys. Rev. D55 (1997) 5112, hep-th/9610043.

[2] L. Motl, "Proposals on nonperturbative superstring interactions", hep-th/9701025.

[3] T. Banks, N. Seiberg, "Strings from Matrices". hep-th/9702187.

[4] R. Dijkgraaf, E. Verlinde, H. Verlinde, "Matrix String Theory", hep-th/9703030.

[5] M. Berkooz, M.R. Douglas, "Five-Branes in Matrix Theory", Phys. Lett. B395 (1997) 196, hep-th/9610236.

[6] T. Banks, N. Seiberg, S. Shenker, "Branes from Matrices", Nucl. Phys. B490 (1997) 91, hep-th/9612157.

[7] M. Berkooz, O. Aharony, "Membrane Dynamics in Matrix Theory", Nucl. Phys. B491 (1997) 184, hep-th/9611215,
G. Lifschytz, S.D. Mathur, "Supersymetry and Membrane Interactions in Matrix Theory", hep-th/9612087,
G. Lifschytz, "Four-Brane and Six-Brane Interaction in Matrix Theory", hep-th/9612223,
V. Balasubramanian, F. Larsen, "Relativistic Brane Scattering", hep-th/9703039,
D. Berenstein, R. Corrado, "Matrix-Theory in Various Dimensions", hep-th/9702108.

[8] O.J. Ganor, S. Ramgoolan, W. Taylor, "Branes, Fluxes and Duality in Matrix-Theory",hep-th/9611202,
L. Susskind, "T Duality in Matrix Theory and S Duality in Field Theory", hep-th/9611164,
W. Taylor, "D-brane field theory on compact spaces", Phys. Lett. B394 (1997) 283, hep-th/9611042.

[9] M.R. Douglas, "Enhanced Gauge Symmetry in Matrix Theory", hep-th/9612126.

[10] W. Fischler, E. Halyo, A. Rajaraman, L. Susskind, "The Incredible Shrinking Torus", hep-th/9703102.

[11] W. Fischler, A. Rajaraman, "Matrix String Theory On K3", hep-th/9704123.

[12] N. Kim, S.-J. Rey, "Matrix Theory on T^5/Z_2 Orbifold and Five-Brane", hep-th/9705132,
N. Kim, S.-J. Rey, "Matrix Theory on an Orbifold and Twisted Membrane", hep-th/9701139.

[13] A. Fayyazuddin, D.J. Smith, "A note on T^5/Z_2 compactification of M-theory matrix model", hep-th/970320.

[14] M. Berkooz, M. Rozali, N. Seiberg, "Matrix Theory Description of M-theory on T^4 and T^5", hep-th/9704089.

[15] N. Seiberg, "New Theories in Six Dimensions and Matrix Description of M-theory on T^5 and T^5/Z_2, hep-th 9705221

[16] S. Govindarajan, “A note on M(atrix) theory in seven dimensions with eight supercharges”, hep-th/9705113.

[17] M. Berkooz, M. Rozali, “String Dualities from Matrix Theory”, hep-th/9705175.

[18] K. Dasgupta, S. Mukhi, “Orbifolds of M-theory”, Nucl. Phys. B465 (1996) 399, hep-th/9512196,
E. Witten, “Five-branes And M-Theory On An Orbifold”, Nucl. Phys. B463 (1996) 383, hep-th/9512219.

[19] H. Ooguri, C. Vafa, “Two-Dimensional Black Hole and Singularities of CY Manifolds”, Nucl. Phys. B463 (1996) 55, hep-th/9511164.

[20] K. Intriligator, N. Seiberg, “Mirror Symmetry in Three Dimensional Gauge Theories”, Phys. Lett. B387 (1996) 513, hep-th/9607207.

[21] J. de Boer, K. Hori, H. Ooguri, Y. Oz, Z. Yin, “Mirror Symmetry in Three-Dimensional Gauge Theories, $SL(2,Z)$ and D-Brane Moduli Spaces”, hep-th/9612131,
J. de Boer, K. Hori, H. Ooguri, Y. Oz, ”Mirror Symmetry in Three-Dimensional Gauge Theories, Quivers and D-branes”,hep-th/9611063.

[22] A. Hanany, E. Witten, “Type IIB Superstrings, BPS Monopoles, And Three-Dimensional Gauge Dynamics”, hep-th/9611230.

[23] M. Porrati, A. Zaffaroni, “M-Theory Origin of Mirror Symmetry in Three Dimensional Gauge Theories”, Nucl. Phys. B490 (1997) 107, hep-th/9611201.

[24] E. Witten, “String Theory In Various Dimensions”, Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[25] P.S. Aspinwall, D.R. Morrison, “ U-Duality and Integral Structures”, Phys. Lett. B355 (1995) 141, hep-th/9507012.

[26] P.S. Aspinwall, “K3 Surfaces and String Duality”, het-th/9611137.

[27] E. Witten, “Some Comments On String Dynamics”, hep-th/9507121.

[28] J. Distler, A. Hanany,”$(0,2)$ Noncritical Strings in Six Dimensions”, Nucl. Phys. B490 (1997) 75, hep-th/9611104.

[29] P.S. Aspinwall, D.R. Morrison, “String Theory on K3 Surfaces”, hep-th/9404151.

[30] P.S. Aspinwall, “Enhanced Gauge Symmetries and K3 Surfaces”, Phys. Lett. B357 (1995) 329, hep-th/9507012.

[31] A. Strominger, “Open P-Branes”, Phys. Lett. B383 (1996) 44, hep-th/9512059.

[32] P.K. Townsend, “Brane Surgery”, Proceedings of the European Research Conference on ’Advanced Quantum Field Theory’. La lonnde les Maures, France, Sept. 1996.

[33] N. Seiberg, “IR Dynamics on Branes and Space-Time Geometry”, Phys.Lett. B384 (1996) 81, hep-th/9606017.

[34] N. Seiberg, E. Witten, “Gauge Dynamics And Compactification To Three Dimensions”, hep-th/9607163.

[35] R.Khuri, “A Multimonopole Solution In String Theory”, Phys. Lett. B294 (1992) 325, hep-th/9205051,

R.Khuri, “A Heterotic Multimonopole Solution”, Nucl. Phys. B387 (1992) 315, hep-th/9205081,
R.Khuri, “Monopoles and Instantons in String Theory”, Phys. Rev. D46 (1992) 4526, hep-th/9205091,
J.P.Gauntlett, J.A. Harvey, J.T. Liu, “Magnetic Monopoles in String Theory”, Nucl. Phys. B409 (1993) 363, hep-th/9211056.

[36] R. Dijkgraaf, E. Verlinde, H. Verlinde, “5D Black Holes and Matrix Strings”, hep-th/9704018.

[37] M. Claudson, M. B. Halpern, “Supersymmetric ground state wave function”, Nucl. Phys. B250 (1985) 689,
R. Flume, “On Quantum Mechanics with extended supersymmetry and nonabelian gauge constraints”, Annals Phys.164 (1985) 189,
M. Baake, M. Reinicke, V. Rittenberg, “Fierz identities for real Clifford algebras and the number of supercharges”, J.Math.Phys.26 (1985) 1070,
V. Rittenberg, S. Yankielowicz, “Supersymmetric gauge theories in quantum mechanics”, Annals Phys.162 (1985), 273.

[38] M.R. Douglas, G. Moore, “ D-branes, Quivers and ALE Instantons”, hep-th/9603167.

[39] P.B. Kronheimer, “The construction of ALE spaces as hyper-kahler quotients”, J. Diff. Geom. 28 (1989) 665.

[40] H. Nakajima, “Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras”, Duke Math. 76 (1994) 365.

[41] M.R. Douglas, B.R. Greene, D.R.Morrison, “Orbifold Resolution by D-Branes”, hep-th/9704151.

[42] J. Polchinski, “Tensors from K3 Orientifolds”, Phys.Rev. D55 (1997) 6423, hep-th/9606165.

[43] D.-E. Diaconescu, M.R. Douglas, J. Gomis, to appear